

Global solution to a cubic nonlinear Dirac equation in $1 + 1$ dimensions

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Abstract

This paper studies a class of nonlinear Dirac equations with cubic terms in R^{1+1} , which include the equations for the massive Thirring model and the massive Gross-Neveu model. Under the assumptions that the initial data has small charge, the global existence of the solution in H^1 are proved. The proof is given by introducing some Bony functional to get the uniform estimates on the nonlinear terms and the uniform bounds on the local smooth solution, which enable us to extend the local solution globally in time. Then L^2 -stability estimates for these solutions are also established by a Lyapunov functional and the global existence of weak solution in L^2 is obtained.

1 Introduction

We consider the nonlinear Dirac equations

$$\begin{cases} i(u_t + u_x) = -mv + N_1(u, v), \\ i(v_t - v_x) = -mu + N_2(u, v), \end{cases} \quad (1.1)$$

with initial data

$$(u, v)|_{t=0} = (u_0(x), v_0(x)) \quad (1.2)$$

where $(t, x) \in R^2$, $(u, v) \in \mathbf{C}^2$, $m \geq 0$. The nonlinear terms satisfy the following:

(A1) N_1 and N_2 have following form:

$$N_1(u, v) = (\alpha_1 u + \alpha_2 \bar{u})(\alpha_3 |v|^2 + \alpha_4 v^2 + \alpha_5 \bar{v}^2),$$

$$N_2(u, v) = (\beta_1 v + \beta_2 \bar{v})(\beta_3 |u|^2 + \beta_4 u^2 + \beta_5 \bar{u}^2),$$

where α_k and β_k ($k = 1, \dots, 5$) are constants.

(A2) For any (u, v) ,

$$\Re(i\bar{u}N_1(u, v) + i\bar{v}N_2(u, v)) = 0.$$

Here and in sequel $\Re Z$ denotes the real part of $Z \in \mathbf{C}$.

Many physical models like massive Thirring model and massive Gross-Neveu model verify (A1) and (A2). Indeed, for massive Thirring model ([23]), we have

$$(N_1, N_2) = \nabla_{(\bar{u}, \bar{v})} \alpha |u|^2 |v|^2 = (\alpha u |v|^2, \alpha v |u|^2), \quad \alpha \in R^1,$$

therefore

$$\Re(i\bar{u}N_1) = \Re(i\bar{v}N_2) = 0.$$

For the massive Gross-Neveu model ([17]), we have

$$(N_1, N_2) = \nabla_{(\bar{u}, \bar{v})} \alpha(\bar{u}v + u\bar{v})^2 = (2\alpha v(\bar{u}v + u\bar{v}), 2\alpha u(\bar{u}v + u\bar{v})), \alpha \in R^1,$$

then

$$\Re(i\bar{u}N_1 + i\bar{v}N_2) = \Re\{2i\alpha(\bar{u}v + u\bar{v})^2\} = 0.$$

The nonlinear Dirac equation is important in quantum mechanics and general relativity ([17] and [23]). There are a number of works on the local and global well-posedness of Cauchy problem in different Sobolev spaces, see for examples [1], [2], [4], [6], [7], [8], [10], [11], [12], [13], [14], [18], [19], [20], [21], [24] and the references therein. The survey of the well-posedness in the nonlinear Dirac equation in one dimension is given in [20]. In this paper we consider a class of cubic nonlinear Dirac equations in one dimension, which include the equation for Massive Gross-Neveu model as an example. To our knowledge, the global existence of solution in H^1 or in L^2 for the Dirac equation of Massive Gross-Neveu model is still open [20]. And it was pointed out by Pelinovsky [20] that the apriori estimates of the L^p -norm $(\|u(t)\|_{L^p}^p + \|v(t)\|_{L^p}^p)^{1/p}$ for the equations like Gross-Neveu model include nonlinear terms, which may lead to the blow-up of the L^∞ and H^1 norms. In addition, the energy of Dirac equation is non-positivity. We will overcome the difficulties caused by the nonlinear terms in the

apriori estimates for $p = 2$ and the non-positivity of the energy by introducing a Bony type functional $Q(t) = \int \int_{x < y} |u(t, x)|^2 |v(t, y)|^2 dx dy$, which is similar to the Glimm's interaction potential for conservation laws ([16]) and is also similar to the Bony functional for Boltzmann equations ([3] and [15]). Then, following Bony's approach [3], for any local H^1 -solution (u, v) with small initial charge $\int_{-\infty}^{\infty} (|u_0|^2 + |v_0|^2) dx$, we can get the control on $\int_0^t \int_{-\infty}^{\infty} |u(x, \tau)|^2 |v(x, \tau)|^2 dx d\tau$ by using the conservation of the charge $\int_{-\infty}^{\infty} (|u(t, x)|^2 + |v(t, x)|^2) dx$; see Lemma 2.2. Here the conservation of charge comes from the assumption (A2) on the nonlinear terms. These bounds enable us to apply the characteristic method to the equations (2.2) to get the uniform L^∞ bounds on $|u|^2$ and $|v|^2$ for extending the local H^1 -solution globally in time; see Lemmas 2.3 and 2.4 for the L^∞ bounds. Then, to study the stability of the H^1 -solutions, we have (3.4) and introduce Lyapunov functional $L_1(t) + KQ_1(t)$ to derive the L^2 estimates on the difference between these H^1 -solutions. Finally global weak solution in L^2 is obtained. We remark that Glimm interaction potential was first used by Glimm [16] to establish the global existence of the BV solution for the system of conservation laws, then was used to study the general conservation laws, and that Bony type functional was used to study Boltzmann equations and wave maps in R^{1+1} , see for instance [16], [5], [9], [22], [3], [15] and [25] and the references therein.

The main results is stated as follows.

Theorem 1.1 *Suppose that $(u_0, v_0) \in H^1(R^1)$ and that the norm $\|u_0\|_{L^2(R^1)}^2 +$*

$\|v_0\|_{L^2(R^1)}^2 \leq C_0$ for some small constant $C_0 > 0$. Then (1.1-1.2) admits a unique global solution $(u, v) \in C([0, \infty); H^1(R^1)) \cap C^1([0, \infty); L^2(R^1))$.

Theorem 1.2 Suppose that $(u_{0k}, v_{0k}) \in H^1(R^1)$ with $\|u_{0k}\|_{L^2(R^1)}^2 + \|v_{0k}\|_{L^2(R^1)}^2 \leq C_0$ ($1 \leq k < \infty$) such that $\|u_{0k} - u_{0\infty}\|_{L^2(R^1)} + \|v_{0k} - v_{0\infty}\|_{L^2(R^1)} \rightarrow 0$ as $k \rightarrow \infty$ for some $(u_{0\infty}, v_{0\infty}) \in L^2(R^1)$. Let (u_k, v_k) be the classical solution of (1.1) ($1 \leq k < \infty$) taking (u_{0k}, v_{0k}) as its initial data. Then there exists a weak solution $(u_\infty, v_\infty) \in C([0, +\infty), L^2(R^2))$ of (1.1) such that

$$\|u_k - u_\infty\|_{C([0, T], L^2(R^2))} + \|v_k - v_\infty\|_{C([0, T], L^2(R^2))} \rightarrow 0$$

as $k \rightarrow \infty$ for any $T \geq 0$.

2 Global classical solution

Since $C_c^\infty(R^1)$ is dense in $H^1(R^1)$, we consider in this section the problem for $(u_0, v_0) \in C_c^\infty(R^1)$. (1.1) is a semilinear hyperbolic system. Local existence of the solutions in H^1 for the semilinear strictly hyperbolic system like (1.1) can be proved by the standard methods using the Duhamel's principle and the fixed point argument, see for instance [2], [10] and [22]. Now we suppose that for $(u_0, v_0) \in C_c^\infty$, the initial value problem (1.1-1.2) has a smooth solution $(u, v) \in C([0, T_*); H^1(R^1)) \cap C^1([0, T_*); L^2(R^1))$ for some $T_* > 0$. Then the support of $(u(x, t), v(x, t))$ is compact for each $t \in [0, T_*)$. To extend the solution across the

time $t = T_*$ we only need to show that

$$\sup_{0 \leq t < T_*} (\|u(t)\|_{H^1} + \|v(t)\|_{H^1}) \leq C'(T_*) \exp(C''(T_*)) \quad (2.1)$$

where $C'(\lambda)$ and $C''(\lambda)$ are polynomials of λ independent of T_* . To this end, we multiply the first equation of (1.1) by \bar{u} and the second equation by \bar{v} , then

$$\begin{cases} (|u|^2)_t + (|u|^2)_x = mi(\bar{u}v - u\bar{v}) + 2\Re(i\bar{N}_1 u), \\ (|v|^2)_t - (|v|^2)_x = -mi(\bar{u}v - u\bar{v}) + 2\Re(i\bar{N}_2 v). \end{cases} \quad (2.2)$$

By Assumption (A2), we have

$$(|u|^2 + |v|^2)_t + (|u|^2 - |v|^2)_x = 0,$$

which leads to the following,

Lemma 2.1 *For any $t \in [0, T_*)$ there holds*

$$\int_{-\infty}^{\infty} (|u(t, x)|^2 + |v(t, x)|^2) dx = \int_{-\infty}^{\infty} (|u_0(x)|^2 + |v_0(x)|^2) dx.$$

Moreover, by Assumption (A1), we can find a constant $c > 0$ such that for any (u, v) there hold the following,

$$|\bar{N}_1 u| + |\bar{N}_2 v| \leq c|u|^2|v|^2. \quad (2.3)$$

Then, (2.2) implies that

$$\begin{cases} (|u|^2)_t + (|u|^2)_x \leq r_0(t, x), \\ (|v|^2)_t - (|v|^2)_x \leq r_0(t, x). \end{cases} \quad (2.4)$$

Here

$$r_0(t, x) = m(|u|^2 + |v|^2)(t, x) + 2c|u(t, x)|^2|v(t, x)|^2.$$

As in [3] (see also [15] and [25]), we define the followings functional for the solution for any $t \in [0, T_*)$,

$$Q_0(t) = \int \int_{x < y} |u(t, x)|^2 |v(t, y)|^2 dx dy,$$

and

$$L_0(t) = \int_{-\infty}^{\infty} (|u(t, x)|^2 + |v(t, x)|^2) dx,$$

$$D_0(t) = \int_{-\infty}^{\infty} |u(t, x)|^2 |v(t, x)|^2 dx.$$

$Q_0(t)$ is called a Bony functional [15] and is similar to the Glimm interaction potential.

Lemma 2.2 *There exists constants $\delta > 0$ such that for the initial data satisfying $L_0(0) \leq \delta$ there holds the following*

$$\frac{dQ_0(t)}{dt} + D_0(t) \leq 2m(L_0(0))^2 \quad (2.5)$$

for $t \in [0, T_*)$. Therefore,

$$Q_0(t) + \int_0^t D_0(\tau) d\tau \leq 2m(L_0(0))^2 t + Q_0(0) \leq 2m(L_0(0))^2 t + (L_0(0))^2 \quad (2.6)$$

for $t \in [0, T_*)$.

Proof. By (2.4), we have

$$\begin{aligned}
\frac{dQ_0(t)}{dt} &\leq - \int \int_{x < y} (|u(t, x)|^2)_x |v(t, y)|^2 dx dy + \int \int_{x < y} |u(t, x)|^2 (|v(t, y)|^2)_y dx dy \\
&\quad + \int \int_{x < y} (r_0(t, x) |v(t, y)|^2 + |u(t, x)|^2 r_0(t, y)) dx dy \\
&\leq -2 \int_{-\infty}^{\infty} |u(t, x)|^2 |v(t, x)|^2 dx \\
&\quad + \int_{-\infty}^{\infty} r_0(t, x) dx \int_{-\infty}^{\infty} |v|^2 dy + \int_{-\infty}^{\infty} |u|^2 dx \int_{-\infty}^{\infty} r_0(t, y) dy \\
&\leq (-2 + 2L_0(t)c) \int_{-\infty}^{\infty} |u(x, t)|^2 |v(x, t)|^2 dx + 2m(L_0(t))^2 \\
&= (-2 + 2L_0(0)c)D_0(t) + 2m(L_0(0))^2,
\end{aligned}$$

where we also use Lemma 2.1 in getting last equality. Now, choose a constant $\delta > 0$ such that

$$-2 + 2\delta c < -1.$$

Then, we can get the desired result. The proof is complete. \square

Next we shall use the above estimates to get the L^∞ bound of the solution for $0 < t < T_*$. Denote

$$\Omega(x_0, t_0) = \{(x, t) : 0 < t < t_0, x_0 - (t_0 - t) < x < x_0 + (t_0 - t)\},$$

and

$$\Gamma_R(x_0, x_0) = \{(x, t) : 0 < t < t_0, x = x_0 + (t_0 - t)\},$$

$$\Gamma_L(x_0, x_0) = \{(x, t) : 0 < t < t_0, x = x_0 - (t_0 - t)\},$$

see Fig. 1.

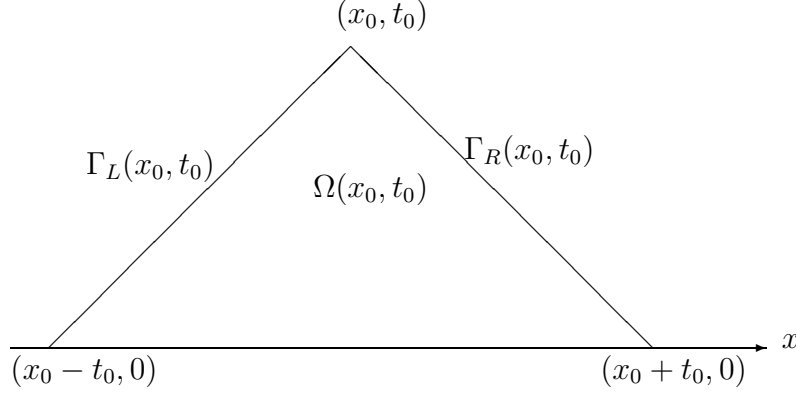


Figure 1: Domain $\Omega(x_0, t_0)$

Lemma 2.3 *For any $t_0 \in (0, T_*)$, there hold*

$$\begin{aligned} \int_{\Gamma_R(x_0, t_0)} |u|^2 &\leq q(t_0), \\ \int_{\Gamma_L(x_0, t_0)} |v|^2 &\leq q(t_0), \end{aligned}$$

where $q(t_0) = t_0(mL_0(0) + 4cm(L_0(0))^2) + 4c(L_0(0))^2 + L_0(0)$.

Proof. Integrating (2.2) over $\Omega(x_0, t_0)$ yields

$$\begin{aligned} \left| \int \int_{\Omega(x_0, t_0)} \left((|u|^2)_t + (|u|^2)_x \right) dx dt \right| &= \int \int_{\Omega(x_0, t_0)} 2|\Re(\bar{u}(miv - iN_1))| dx dt \\ &\leq \int \int_{\Omega(x_0, t_0)} (m(|u|^2 + |v|^2) + 2c|u|^2|v|^2) dx dt \\ &\leq mt_0L_0(0) + 2c \int_0^{t_0} D(\tau) d\tau \\ &\leq t_0(mL_0(0) + 4cm(L_0(0))^2) + 4c(L_0(0))^2 \end{aligned}$$

and

$$\begin{aligned}
\left| \int \int_{\Omega(x_0, t_0)} \left((|v|^2)_t - (|v|^2)_x \right) dx dt \right| &= 2 |\Re(\bar{v}(m i u - i N_2))| \\
&\leq m t_0 L_0(0) + 2c \int_0^{t_0} D(\tau) d\tau \\
&\leq t_0 (m L_0(0) + 4cm(L_0(0))^2) + 4c(L_0(0))^2,
\end{aligned}$$

where we use (2.3) and use Lemmas 2.1 and 2.2. Then the desired estimates follow by applying Green formula to the above. The proof is complete. \square

Lemma 2.4 *For $t \in [0, T_*)$, there holds the following,*

$$\sup_{x \in R^1} (|u(x, t)|^2 + |v(x, t)|^2) \leq \left(\sup_{x \in R^1} (|u_0(x)|^2 + |v_0(x)|^2) + 2mq(T_*) \right) \exp(mT_* + 2cq(T_*)),$$

where $q(t)$ is given in Lemma 2.3.

Proof. Using the characteristic method, by (2.4) we have

$$\frac{d}{dt} |u(t, x_0 + (t - t_0))|^2 \leq \left((m + 2c|v|^2)|u|^2 + m|v|^2 \right) \Big|_{(t, x_0 + (t - t_0))}$$

and

$$\frac{d}{dt} |v(t, x_0 - (t - t_0))|^2 \leq \left((m + 2c|u|^2)|v|^2 + m|u|^2 \right) \Big|_{(t, x_0 - (t - t_0))},$$

for any $x_0 \in R^1$, $t_0 \in [0, T_*)$. Then,

$$\begin{aligned}
|u(t_0, x_0)|^2 &\leq (|u_0(x_0 - t_0)|^2 + m \int_0^{t_0} |v(\tau, x_0 + (\tau - t_0))|^2 d\tau) h_1(t_0), \\
|v(t_0, x_0)|^2 &\leq (|v_0(x_0 + t_0)|^2 + m \int_0^{t_0} |u(\tau, x_0 - (\tau - t_0))|^2 d\tau) h_2(t_0),
\end{aligned}$$

where

$$\begin{aligned} h_1(t_0) &= \exp \left(mt_0 + 2c \int_0^{t_0} |v(x_0 + (\tau - t_0), \tau)|^2 d\tau \right), \\ h_2(t_0) &= \exp \left(mt_0 + 2c \int_0^{t_0} |u(x_0 - (\tau - t_0), \tau)|^2 d\tau \right). \end{aligned}$$

Therefore, these lead to the desired result by Lemma 2.3. The proof is complete.

□

Proof of Theorem 1.1. It suffices to get the H^1 bound of the solution on $[0, T_*)$. We differentiate Eqs. 1.1 with respect to x ; then

$$\begin{cases} i(u_{tx} + u_{xx}) = -mv_x + (N_1(u, v))_x, \\ i(v_{tx} - v_{xx}) = -mu_x + (N_2(u, v))_x. \end{cases} \quad (2.7)$$

For the nonlinear terms we have the following estimates by Lemma 2.4:

$$\|(N_1)_x\|_{L^2(R^1)}^2 + \|(N_2)_x\|_{L^2(R^1)}^2 \leq C_1 q_1(T_*) (\|u_x(t, \cdot)\|_{L^2(R^1)}^2 + \|v_x(t, \cdot)\|_{L^2(R^1)}^2)$$

for $t \in [0, T_*)$. Here C_1 is a constant depending on m and $\alpha_j, \beta_j, 1 \leq j \leq 5$; $q_1(T_*) = (1 + q(T_*))^2 \exp(2mT_* + 2cq(T_*))$. This estimates, together with the energy method, we can get the desired bound as (2.1). Thus we can extend the solution across $t = T_*$. The proof is complete. □

3 Convergence in L^2

First we establish the estimates on the difference of smooth solutions. Let (u', v') be the global smooth solution to (1.1) taking $(u'_0, v'_0) \in C_c^\infty(R^1)$ as its initial data with $\int_{-\infty}^{\infty} (|u'_0(x)|^2 + |v'_0(x)|^2) dx < \delta$.

Let $U = u - u'$ and $V = v - v'$. Then,

$$\begin{cases} U_t + U_x = imV - i(N_1(u, v) - N_1(u', v')), \\ V_t - V_x = imU - i(N_2(u, v) - N_2(u', v')), \end{cases} \quad (3.1)$$

which leads to

$$\begin{cases} (|U|^2)_t + (|U|^2)_x = \Re 2\{imV\bar{U} - i(N_1(u, v) - N_1(u', v'))\bar{U}\}, \\ (|V|^2)_t - (|V|^2)_x = \Re 2\{imU\bar{V} - i(N_2(u, v) - N_2(u', v'))\bar{V}\} \end{cases} \quad (3.2)$$

and

$$(|U|^2 + |V|^2)_t + (|U|^2 - |V|^2)_x = -R \quad (3.3)$$

with

$$R = \Re\{2i(N_1(u, v) - N_1(u', v'))\bar{U} + 2i(N_2(u, v) - N_2(u', v'))\bar{V}\}.$$

For the nonlinear terms in righthand side in the above, we have the following.

Lemma 3.1 *There exists a constant $c_* > 0$ such that*

$$|\Re 2\{imV\bar{U} - i(N_1(u, v) - N_1(u', v'))\bar{U}\}| \leq r_1(t, x),$$

$$|\Re 2\{imU\bar{V} - i(N_2(u, v) - N_2(u', v'))\bar{V}\}| \leq r_1(t, x),$$

and

$$|R| \leq c_* r_2(t, x, x),$$

where $r_1(t, x) = m(|U(t, x)|^2 + |V(t, x)|^2) + c_* r_2(t, x, x)$, and

$$r_2(t, x, y) = |U(t, x)|^2(|v(t, y)|^2 + |v'(t, y)|^2) + (|u(t, x)|^2 + |u'(t, x)|^2)|V(t, y)|^2.$$

Proof. Assumption (A1) implies that there exists a constant $c_* > 0$ such that

$$\begin{aligned}
|(N_1(u, v) - N_1(u', v'))\overline{U}| &= |N_1(u, v) - N_1(u', v')||U| \\
&\leq \frac{c_*}{8}(|u - u'||v|^2 + |u'||v - v'||v| + |u'||v'||v - v'|)|U| \\
&\leq \frac{c_*}{4}r_2(t, x, x),
\end{aligned}$$

and

$$|(N_2(u, v) - N_2(u', v'))\overline{V}| \leq \frac{c_*}{4}r_2(t, x, x).$$

Thus, the above estimates give the desired results. The proof is complete. \square

Now it follows from (3.2) that

$$\begin{cases} (|U|^2)_t + (|U|^2)_x \leq r_1(t, x), \\ (|V|^2)_t - (|V|^2)_x \leq r_1(t, x). \end{cases} \quad (3.4)$$

And to deal with the nonlinear terms in (3.1), we define the following for any

$t \geq 0$,

$$L_1(t) = \int_{-\infty}^{\infty} (|U(t, x)|^2 + |V(t, x)|^2) dx,$$

$$Q_1(t) = \int \int_{x < y} r_2(t, x, y) dx dy,$$

$$D_1(t) = \int_{-\infty}^{\infty} r_2(t, x, x) dx.$$

Let

$$r'_0(t, x) = m(|u'(t, x)|^2 + |v'(t, x)|^2) + c|u'(t, x)|^2|v'(t, x)|^2,$$

and

$$Q'_0(t) = \int \int_{x < y} |u'(t, x)|^2 |v'(t, y)|^2 dx dy,$$

and

$$L_0(t) = \int_{-\infty}^{\infty} (|u'(t, x)|^2 + |v'(t, x)|^2) dx,$$

$$D_0(t) = \int_{-\infty}^{\infty} |u'(t, x)|^2 |v'(t, x)|^2 dx.$$

Lemma 3.2 *There exist constants $\delta > 0$ and $K > 0$ such that if $L_0(0) < \delta$ and $L'_0(0) < \delta$ then*

$$\frac{d}{dt}(L_1(t) + KQ_1(t)) + D_1(t) \leq (2mL_0(0) + 2mL'_0(0) + cD_0(t) + cD'_0(t))L_1(t) \quad (3.5)$$

for $t \geq 0$. Therefore

$$L_1(t) + KQ_1(t) \leq (L_1(0) + KQ_1(0)) \exp(h_3(t)) \quad (3.6)$$

and

$$\int_0^t D_1(\tau) d\tau \leq (L_1(0) + KQ_1(0))(1 + (4m\delta + \delta^2 + 2m\delta^2 t) \int_0^t \exp(h_3(\tau)) d\tau) \quad (3.7)$$

for $t \geq 0$, where

$$h_3(t) = 2mL_0(0)t + 2mL'_0(0)t + \int_0^t (cD_0(\tau) + cD'_0(\tau)) d\tau.$$

Proof. (3.4) yields that

$$\frac{d}{dt}L_1(t) \leq 2c_*D_1(t),$$

and

$$\frac{d}{dt}Q_1(t) \leq -2D_1(t) + \int \int_{x < y} r_1(t, x) (|v(t, y)|^2 + |v'(t, y)|^2) dx dy$$

$$\begin{aligned}
& + \int \int_{x < y} |U(t, x)|^2 (r_0(t, y) + r'_0(t, y)) dx dy \\
& + \int \int_{x < y} (|u(t, x)|^2 + |u'(t, x)|^2) r_1(t, x) dx dy \\
& + \int \int_{x < y} (r_0(t, x) + r'_0(t, x)) |V(t, y)|^2 dx dy \\
\leq & -2D_1(t) + (mL_1(t) + c_* D_1(t))(L_0(t) + L'_0(t)) \\
& + (mL_0(t) + mL'_0(t) + cD_0(t) + cD'_0(t))L_1(t) \\
\leq & \left(-2 + c_*(L_0(0) + L'_0(0)) \right) D_1(t) \\
& + \left(2mL_0(0) + 2mL'_0(0) + cD_0(t) + cD'_0(t) \right) L_1(t),
\end{aligned}$$

where Lemma 2.1 is used in the last inequality. These estimates give (3.5) for small $L_0(0) < \delta$ and $L'_0(0) < \delta$ and for Large K , where $\delta = \frac{1}{4c_*}$. (3.6) is a consequence of (3.5).

To prove (3.7), we integrate (3.5) over $[0, t]$, then

$$\begin{aligned}
\int_0^t D_1(\tau) d\tau & \leq L_1(0) + KQ_1(0) + \int_0^t (4m\delta + cD_0(\tau) + cD'_0(\tau)) L_1(\tau) d\tau \\
& \leq (L_1(0) + KQ_1(0)) + (4m\delta + \delta^2 + 2m\delta^2 t) \int_0^t L_1(\tau) d\tau \\
& \leq (L_1(0) + KQ_1(0)) (1 + (4m\delta + \delta^2 + 2m\delta^2 t) \int_0^t \exp(h_3(\tau)) d\tau),
\end{aligned}$$

where Lemma 2.2 and (3.6) are used. The proof is complete. \square

Lemma 3.3 *Let $h_3(t)$ be given by Lemma 3.2. Then for $t \geq 0$,*

$$h_3(t) \leq 2m \left(L_0(0) + L'_0(0) + (L_0(0))^2 + (L'_0(0))^2 \right) t + (L_0(0))^2 + (L'_0(0))^2.$$

Proposition 3.1 *Let (u, v) and (u', v') be two classical solutions to 1.1 with initial data (u_0, v_0) and (u'_0, v'_0) respectively, and suppose that $\|u_0\|_{L^2(R^1)}^2 + \|v_0\|_{L^2(R^1)}^2 < \delta$ and $\|u'_0\|_{L^2(R^1)}^2 + \|v'_0\|_{L^2(R^1)}^2 < \delta$. Then, there exist constants, c_1, c_2 and c_3 , depending on the δ , such that for $t \geq 0$,*

$$\|u(t) - u'(t)\|_{L^2(R^1)}^2 + \|v(t) - v'(t)\|_{L^2(R^1)}^2 \leq (\|u_0 - u'_0\|_{L^2(R^1)}^2 + \|v_0 - v'_0\|_{L^2(R^1)}^2)h_4(t),$$

where $h_4(t) = c_1 \exp(c_2 t + c_3)$.

Proof. Indeed, we have

$$Q_1(0) \leq \int_{-\infty}^{\infty} (|U(0, x)|^2 + |V(0, x)|^2) dx (L_0(0) + L'_0(0)),$$

which, together with (3.6), gives the proof of this proposition. Thus the proof is complete. \square

Proof of Theorem 1.2 By Proposition 3.1, we have

$$\begin{aligned} & \|u_k(t) - u_j(t)\|_{L^2(R^1)}^2 + \|v_k(t) - v_j(t)\|_{L^2(R^1)}^2 \\ & \leq (\|u_{k0} - u_{j0}\|_{L^2(R^1)}^2 + \|v_{k0} - v_{j0}\|_{L^2(R^1)}^2)h_4(t), \end{aligned}$$

which implies that there exists a unique $(u_\infty, v_\infty) \in C([0, +\infty), L(R^2))$ such that

$$\|u_k - u_\infty\|_{C([0, T], L(R^2))} + \|v_k - v_\infty\|_{C([0, T], L(R^2))} \rightarrow 0$$

for any $T \geq 0$.

Now to prove that (u_∞, v_∞) is a weak solution of (1.1), let

$$U_{k,j} = u_k - u_j, \quad V_{k,j} = v_k - v_j,$$

and

$$r_{k,j} = |U_{k,j}(t, x)|^2 (|v_k(t, x)|^2 + |v_j(t, x)|^2) + (|u_k(t, x)|^2 + |u_j(t, x)|^2) |V_{k,j}(t, x)|^2.$$

By (3.7), for any $T > 0$, there exists a constant $C_1(T) > 0$ such that for $l = 1, 2$

$$\begin{aligned} \int_{-\infty}^{\infty} |N_l(u_k, v_k) - N_l(u_j, v_j)| dx &\leq c_* \int_{-\infty}^{\infty} r_{k,j} dx \\ &\leq C_1(T) c_* (\|U_{k,j}(t)\|_{L^2(R^1)}^2 + \|V_{k,j}(t)\|_{L^2(R^1)}^2) \\ &\leq C_1(T) c_* (\|U_{k,j}(0)\|_{L^2(R^1)}^2 + \|V_{k,j}(0)\|_{L^2(R^1)}^2), \end{aligned}$$

which leads to the strong convergence of the nonlinear terms. Therefore, (u_∞, v_∞)

is a weak solution of (1.1). The proof of Theorem 1.2 is complete. \square

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